1) Chapter :- Relation between roots and Coefficient

By Prof. Abhimanyu Kumar Choudhary
H.O.D. of Math, B.N. College, Bhagalpur

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Date

(1) Find the condition that the cubic equation $x^3 - px^2 + qx - r = 0$ should have its roots in H.P.

Soln → Let the roots of cubic be α, β, γ then

$\alpha + \beta + \gamma = p$ —— (i)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ —— (ii)

and $\alpha\beta\gamma = r$ —— (iii)

Given that α, β, γ are in H.P.

$\therefore \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P.

$\therefore \frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta} \Rightarrow \frac{\gamma + \alpha}{\alpha\gamma} = \frac{2}{\beta}$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = 3\gamma\alpha$

$\Rightarrow q = 3\gamma\alpha$ (from (ii))

$\Rightarrow q = \frac{3r}{\beta}$ [from (iii)]

$\therefore \beta = \frac{3r}{q}$

$\because \beta$ is a root of the equation

$\therefore \beta^3 - p\beta^2 + q\beta - r = 0$

$\Rightarrow \left(\frac{3r}{q}\right)^3 - p\left(\frac{3r}{q}\right)^2 + q\left(\frac{3r}{q}\right) - r = 0$

$\Rightarrow \frac{27r^3}{q^3} - \frac{9r^2p}{q^2} + 3r - r = 0$

$\Rightarrow 27r^3 - 9pqr^2 + 2rq^3 = 0 \Rightarrow r(27r^2 - 9pqr + 2q^3) =$

$\Rightarrow 27r^2 - 9pqr + 2q^3 = 0$

Which is the required condition

②

B.Sc. part I

Find the condition that the roots of the equation $ax^3+3bx^2+3cx+d=0$ are in (I) A.P. (II) G.P. (III) H.P.

Sol^n →

(I) Let $\alpha-\delta, \alpha, \alpha+\delta$ be the roots of the given eq^n

then $(\alpha-\delta)+\alpha+(\alpha+\delta) = -\frac{3b}{a}$

$\Rightarrow 3\alpha = -\frac{3b}{a}$ $\therefore \alpha = -\frac{b}{a}$ —— (1)

$\because$ α is a root of the equation $ax^3+3bx^2+3cx+d=0$ then we have

$a\alpha^3+3b\alpha^2+3c\alpha+d=0$

$a\left(-\frac{b}{a}\right)^3+3b\left(-\frac{b}{a}\right)^2+3c\left(-\frac{b}{a}\right)+d=0$

$\Rightarrow -a\cdot\frac{b^3}{a^3}+3b\cdot\frac{b^2}{a^2}-\frac{3bc}{a}+d=0$

$\Rightarrow -\frac{b^3}{a^2}+\frac{3b^3}{a^2}-\frac{3bc}{a}+d=0$

$\Rightarrow \frac{2b^3}{a^2}-\frac{3bc}{a}+d=0$

$\therefore 2b^3-3abc+a^2d=0$

Which is the required condition.

(II)

$\because \alpha\beta\gamma = -\frac{d}{a}$ —— (1)

$\because \alpha, \beta, \gamma$ are in G.P. then

Let $\alpha = \frac{\lambda}{\gamma}, \beta = \lambda$ and $\gamma = \lambda\gamma$

$\therefore \frac{\lambda}{\gamma}\times\lambda\times\lambda\gamma = -\frac{d}{a} \Rightarrow \lambda^3 = -\frac{d}{a}$ —— (2)

$\because \lambda(=\beta)$ is a root of the given equ^n

$\therefore$ we have $a\lambda^3+3b\lambda^2+3c\lambda+d=0$

$\Rightarrow a\left(-\frac{d}{a}\right) + 3b\lambda^2 + 3c\lambda + d = 0$

$\Rightarrow -d + 3b\lambda^2 + 3c\lambda + d = 0$ $\Rightarrow b\lambda + c = 0$

$\Rightarrow 3b\lambda^2 + 3c\lambda = 0$ $\Rightarrow b^3\lambda^3 = -c^3$

$\Rightarrow b\lambda = -c \Rightarrow b^3\lambda^3 = -c^3$ $\Rightarrow b^3 d = ac^3$

$\Rightarrow b^3\left(-\frac{d}{a}\right) = -c^3$

$\Rightarrow b^3 d - c^3 a = 0$

which is the required condition.

Let α, β, γ be the roots of given equation then $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{3c}{a}$ — (ii)

$\alpha + \beta + \gamma = \frac{-3b}{a}$ — (i)

$\alpha\beta\gamma = -d/a$ — (iii)

$\because \alpha, \beta, \gamma$ are in HP $\therefore \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in AP

$\therefore \frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta} \Rightarrow \frac{\alpha + \gamma}{\alpha\gamma} = \frac{2}{\beta}$

i.e. $\alpha\beta + \beta\gamma = 2\gamma\alpha \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = 3\gamma\alpha$.

$\Rightarrow \frac{3c}{a} = 3\gamma\alpha$ (from (2)) $\Rightarrow \gamma\alpha = \frac{c}{a}$

Hence from (iii) $\beta \cdot \frac{c}{a} = -\frac{d}{a}$ $\therefore \beta = \frac{-d}{c}$

$\because \beta$ is the roots of equ^n $ax^3 + 3bx^2 + 3cx + d = 0$

$\therefore a\beta^3 + 3b\beta^2 + 3c\beta + d = 0$

$\Rightarrow a\left(-\frac{d}{c}\right)^3 + 3b\left(-\frac{d}{c}\right)^2 + 3c\left(\frac{-d}{c}\right) + d = 0$

$\Rightarrow -a\frac{d^3}{c^3} + 3\frac{bd^2}{c^2} - 3d + d = 0$

$\Rightarrow -\frac{ad^3}{c^3} + \frac{3bd^2}{c^2} - 2d = 0$

$\Rightarrow -ad^3 + 3bcd^2 - 2c^3 d = 0 \Rightarrow ad^2 - 3bcd + 2c^3 = 0$

which is the required condition

(4) Find the condition that the equation $x^4+px^3+qx^2+rx+s=0$ should have two roots connected by the relation $\alpha+\beta=0$. Determine in that case two quadratic equation which shall have for roots (i) α,β and (ii) γ,δ.

Soln →

Let $\alpha,\beta,\gamma,\delta$ be the roots of the equation

$$x^4+px^3+qx^2+rx+s=0$$

$$\therefore\ \alpha+\beta+\gamma+\delta=-p \quad \text{--- (i)}$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q \quad \text{--- (ii)}$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r \quad \text{--- (iii)}$$

$$\alpha\beta\gamma\delta=s \quad \text{--- (iv)}$$

Also $\alpha+\beta=0$ --- (v)

from (i) & (v) $\gamma+\delta=-p$

from (iii) $\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-r$

$$\Rightarrow \alpha\beta(-p)+\gamma\delta\cdot 0=-r$$

$$\Rightarrow \alpha\beta p=r \Rightarrow \alpha\beta=\frac{r}{p}$$

from (iv) $\gamma\delta=\frac{s}{\alpha\beta}=\frac{ps}{r}$

from (ii) $\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta)=q$

$$\Rightarrow \frac{r}{p}+\frac{ps}{r}+0\cdot(-p)=q$$

$$\Rightarrow \frac{r^2+p^2s}{pr}=q \Rightarrow r^2+p^2s=pqr$$

which is required condition.

For (ii) → we have $\alpha+\beta = 0, \alpha\beta = \frac{\gamma}{\beta}$

∴ the equation whose roots are (α, β) is

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

$$\Rightarrow x^2 - 0 \cdot x + \frac{\gamma}{\beta} = 0$$

$$\Rightarrow \beta x^2 + \gamma = 0$$

Also we have, $\gamma+\delta = -\beta$ and $\gamma\delta = \frac{\beta\delta}{\gamma}$

Hence the equation whose roots are (γ, δ) is

$$x^2 - (\gamma+\delta)x + \gamma\delta = 0$$

$$\Rightarrow x^2 - (-\beta)x + \frac{\beta\delta}{\gamma} = 0$$

ie, $\gamma x^2 + \beta\gamma x + \beta\delta = 0$