B.Sc. part II (Differential Calculus)

Continuity & Derivability

By – Prof. A.K. Choudhary.
Head of Dept. of Math, B.N. College Bhagalpur.

Theorem → If a function defined in a closed interval $[a, b]$ possesses a finite derivative at a point of the interval, then $f(x)$ is also continuous at that point.

Proof →

Let $f(x)$ have a finite derivative $f'(c)$ at $x = c$ where $c \in [a, b]$. Then

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c).$$

Now we observe that

$$f(c+h) - f(c) = \frac{f(c+h) - f(c)}{h} \cdot h.$$

$$\therefore \lim_{h \to 0} [f(c+h) - f(c)] = \lim_{h \to 0} \left\{ \frac{f(c+h) - f(c)}{h} \cdot h \right\}$$

$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \times \lim_{h \to 0} h$$

$$= f'(c) \times 0$$

$$= 0 \quad [\because f'(c) \text{ is finite}]$$

Thus we have $\lim_{h \to 0} f(c+h) = f(c)$

Which $\Rightarrow$ that $f(x)$ is continuous at $x = c$ where $c \in [a, b]$.

Proved

Ex-1 Prove that the function $f(x) = |x|$ is Continuous at $x=0$ but not differentiable at $x=0$.

Soln → For continuity of the function at $x=0$ we have $f(0)=0$.

$$f(0+0) = \underset{h\to 0}{Lt} f(0+h) = \underset{h\to 0}{Lt} |0+h| = 0$$

$$f(0-0) = \underset{h\to 0}{Lt} f(0-h) = \underset{h\to 0}{Lt} |0-h| = 0$$

$$\therefore f(0+0) = f(0-0) = f(0).$$

$\therefore f(x)$ is continuous at $x=0$.

For differentiability, we have

$$Rf'(0) = \underset{h\to 0}{Lt} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0}{Lt} \frac{|h|-0}{h} = 1$$

$$Lf'(0) = \underset{h\to 0}{Lt} \frac{f(0-h)-f(0)}{-h} = \underset{h\to 0}{Lt} \frac{|-h|-0}{-h}$$

$$= \underset{h\to 0}{Lt} \frac{h}{-h} = -1$$

$$\therefore Rf'(0) = Lf'(0).$$

$\therefore f(x)$ is not differentiable at $x=0$.

Ex-2 Discuss the continuity and differentiability of the following function at $x=0$

$$f(x) = 2+x \text{ if } x \geq 0$$
$$= 2-x \text{ if } x < 0.$$

Soln → For continuity, we have

$$f(0+0) = \underset{h\to 0}{Lt} f(0+h) = \underset{h\to 0}{Lt} f(h) = \underset{h\to 0}{Lt} f(2+h) = 2$$

(3) $f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}(2-h) = 2$

$\because f(0) = 2$

$\therefore f(0+0) = f(0-0) = f(0)$

$\therefore f(x)$ is continuous at $x=0$

For differentiability, we have

$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{(2+h)-2}{h} = 1$

$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$

$= \lim_{h\to 0} \frac{(2+h)-2}{-h} = -1$

$\because Rf'(0) \neq Lf'(0)$.

$\therefore f(x)$ is not differential at $x=0$.

Ex (3)

Examine the continuity and differentiability of the function

$f(x) = x \sin\frac{1}{x} \quad x \neq 0$

$f(0) = 0 \quad$ at the point $x=0$

Sol^n →

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) \quad \left[\because \sin\frac{1}{h} \le 1\right]$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} -h\sin\frac{1}{-h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

$\therefore f(0) = f(0+0) = f(0-0)$

$\therefore f(x)$ is continuous at $x=0$.

For differentiable

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} \sin\frac{1}{h}$$

which does not exists

Similarly $Lf'(0)$ which does not exists.

Hence $f(x)$ is not differentiable at $x=0$.

Ex (5) If $f(x) = x^2 \sin\frac{1}{x}$, when $x \ne 0$

$f(0) = 0$ show that $f(x)$ is continuous and differentiable at $x=0$.

Soln. $\rightarrow$ For continuity

$\therefore f(0) = 0$

(5) $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0 \quad [\because \sin \frac{1}{h} \leq 1]$

and $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0} h^2 \sin \frac{1}{-h} = 0$

$\therefore f(0) = f(0+0) = f(0-h).$

$\therefore f(x)$ is continuous at $x=0$.

For differentiability

$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\cancel{2}} \sin \frac{1}{h}}{\cancel{h}} = 0$

and $Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{-h}}{-h}$

$= \lim_{h \to 0} \frac{h^{\cancel{2}} \sin \frac{1}{h}}{\cancel{h}} = 0$

Hence $Rf'(0) = Lf'(0)$

$\therefore f(x)$ is differentiable at $x=0$.